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Fractal dimension and degeneracy of the critical point for iterated maps

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Abstract. Variations of the three most commonly used definitions of fractal dimension, namely the capacity, the information dimension and the correlation exponent, with the degeneracy of the critical point is studied both numerically and analytically. The numerical results agree very well with analytical estimates. It is found that these three fractal dimensions show quite different behaviour as the degeneracy of the critical point is varied. Therefore, although they are sometimes used almost interchangeably as measures of fractal dimension, they are actually very different concepts.

1. Introduction

Since Ruelle and Takens (1971) proposed the idea of a strange attractor as a possible agent responsible for the occurrence of turbulence, this scenario has attracted considerable interest. However, it was not until quite recently when experimental evidence (Abraham *et al* 1984) for the existence of strange attractors began to accumulate that credibility was lent to this previously purely theoretical concept.

To quantify a strange attractor, the most commonly used measures are the fractal dimension, the Lyapunov exponent and the metric entropy. There is a proliferation of definitions of fractal dimensions (Farmer *et al* 1983). However, the three most commonly employed are the capacity D, the information dimension σ and the correlation exponent ν. The capacity D is determined by the box-counting algorithm:

$$D = \lim_{\varepsilon \to 0} \frac{\log M(\varepsilon)}{\log(1/\varepsilon)} \tag{1}$$

where $M(\varepsilon)$ is the number of non-empty hypercubes of side ε needed to cover the attractor. Since the definition of D is purely geometric in nature, it is oblivious to the frequency at which various parts of the attractor are visited. The notion of the information dimension was introduced to take into account the possible non-uniform distribution of points on the attractor. It is defined as

$$\sigma = \lim_{\varepsilon \to 0} \frac{S(\varepsilon)}{\log(1/\varepsilon)} \tag{2}$$

where p_i is the probability for a point to fall into the ith box, and $S(\varepsilon) = -\sum_{i=1}^{M(\varepsilon)} p_i \log p_i$ is the information entropy. The box-counting algorithm requires an enormous amount of memory and becomes prohibitively time-consuming as the dimension increases.

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The correlation exponent ν was introduced to by-pass such difficulties (Grassberger and Procaccia 1983a, b). It is defined as

$$\nu = \lim_{\varepsilon \to 0} \frac{\log C(\varepsilon)}{\log \varepsilon} \tag{3}$$

where the correlation sum $C(\varepsilon)$ is

$$C(\varepsilon) = \lim_{n \to \infty} \frac{1}{n^2} \sum_{i \neq j}^{n} \theta(\varepsilon - |x_i - x_j|) \tag{4}$$

and $\theta(x)$ is the Heaviside step function.

There is some indication (Grassberger 1981) that the fractal dimension is also a universal number as the period-doubling bifurcation ratios α and δ. However, as we know, for iterated maps of the interval

$$f(x) = 1 - a|x|^z \tag{5}$$

α and δ actually depend on the degeneracy z of the critical point (Hu and Mao 1982, Hu 1982, Hu and Satija 1983). It is interesting to ask how the various fractal dimensions will then depend on z. We have therefore calculated D, σ and ν for $z = 2, 3, 4, 5$ at the period-doubling accumulation point a^*. In § 2 the numerical results are presented. Approximate analytical formulae for these fractal dimensions are derived in § 3. Finally, in § 4, some concluding remarks are given.

2. Numerical results

Numerical results for the capacity D, the information dimension σ and the correlation experiment ν are presented as a log-log plot in figures 1, 2 and 3 respectively. The power-law behaviour is seen to be well obeyed.

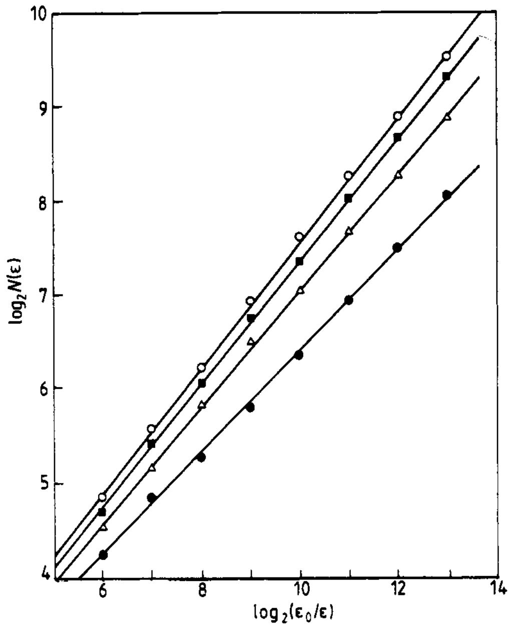

Figure 1. The capacity D for the critical maps $f(x) = 1 - a^*|x|^z$, $z = 2$ (●), 3 (△), 4 (■) and 5 (○).

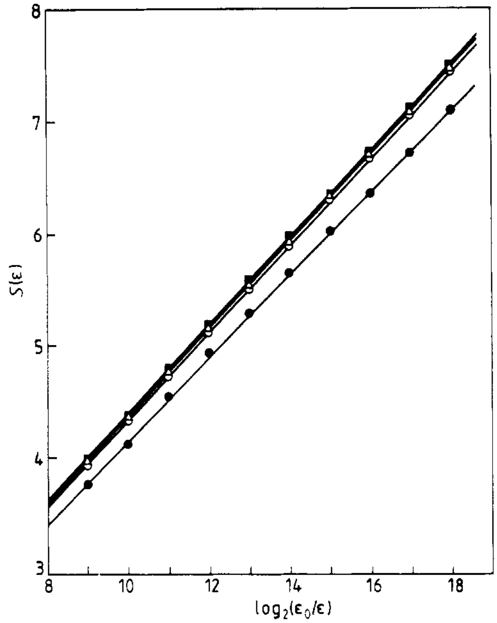

Figure 2. The information dimension σ for the critical maps $(x) = 1 - a^*|x|^z$, $z = 2$ (●), 3 (△), 4 (■) and 5 (○).

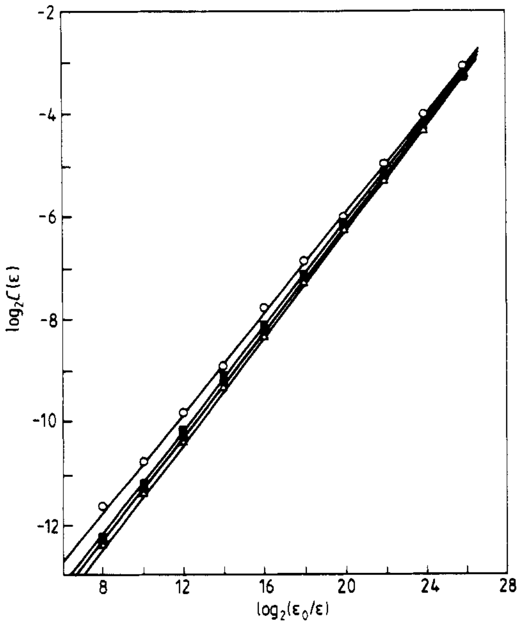

Figure 3. The correlation exponent ν for critical maps $f(x) = 1 = a^*|x|^z$, $z = 2$ (●), 3 (△), 4 (■) and 5 (○).

Table 1. The capacity D, the information dimension σ and the correlation exponent ν for the critical maps $f(x) = 1 - a^*|x|^z$, $z = 2, 3, 4, 5$. Both the numerical and analytical values are listed.

	D		σ		ν	
z	Numerical	Analytical	Numerical	Analytical	Numerical	Analytical
2	0.538 ± 0.01	0.538	0.517 ± 0.08	0.517	0.501 ± 0.02	0.499
3	0.601 ± 0.01	0.606	0.557 ± 0.01	0.551	0.509 ± 0.02	0.509
4	0.640 ± 0.09	0.642	0.556 ± 0.04	0.555	0.498 ± 0.05	0.497
5	0.661 ± 0.08	0.665	0.558 ± 0.02	0.550	0.483 ± 0.03	0.481

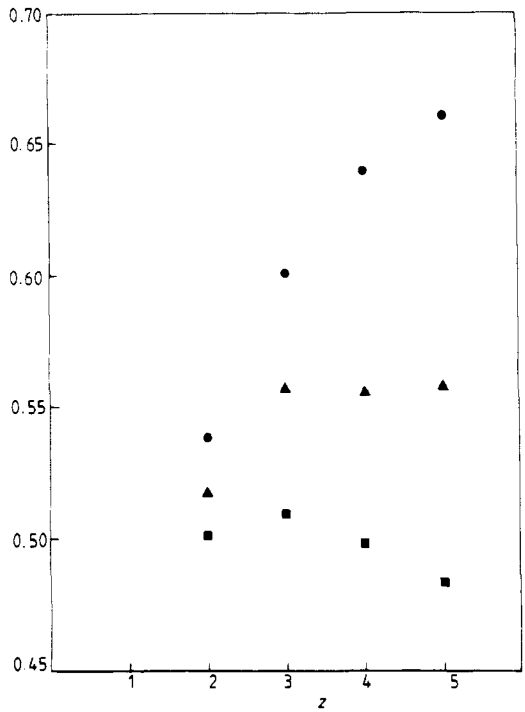

Figure 4. The capacity D (●), the information dimension σ (▲) and the correlation exponent ν (■) as a function of the degeneracy z of the critical point.

The values for D, σ and ν are given by the slopes of the straight lines. They are listed in table 1 and plotted in figure 4. We have used the method of least-square fits to compute the slopes and their errors. In general (with one exception: $\sigma(z = 2)$, see table 1) the error bars are bigger for higher z because higher powers introduce bigger errors when iterating the map to higher orders.

3. Analytical approximations for the fractal dimensions

We will now show in detail how to derive the analytical approximations for the fractal dimensions (Grassberger 1981, Hentschel and Procaccia 1983). The derivation of these formulae is based on the self-similarity of the Feigenbaum strange attractor. The Feigenbaum attractor consists of a set of points $\{x_i\}^k = \{x_i, i = 1, \ldots, 2^k\}$ generated by the map $x_{i+1} = g(x_i)$ where $g(x)$ is the invariant function satisfying the functional equation

$$g(g(x)) = -\alpha^{-1} g(\alpha x) \tag{6}$$

with the boundary condition $g(0) = 1$.

We sketch the set in figure 5(a) for $k = 2$, 3 and 4, taking $x_0 = 0$. In figure 5(b), we have rescaled the set such that $x_2^{(b)} = 0$ and $x_1^{(b)} = 1$, i.e.

$$x_1^{(b)} = \varepsilon_0^{-1}\left(x_1^{(a)} + \frac{1}{\alpha}\right) \tag{7a}$$

where

$$\varepsilon_0 = 1 + 1/\alpha. \tag{7b}$$

The set $\{x_1\}^k$ consists of two subsets: the even subset and the odd subset,

$$\{x_i\}^k = \{x_i\}^k_{\text{even}} + \{x_i\}^k_{\text{odd}} \tag{8a}$$

where

$$\{x_i\}^k_{\text{even}} = \{x_1, i = 2, 4, \ldots, 2^k\} \tag{8b}$$

and

$$\{x_i\}^k_{\text{even}} = \{x_1, i = 1, 3, \ldots, 2^k - 1\}. \tag{8c}$$

We will show that the even subset is exactly similar to the whole set. To see this, let us first prove inductively that

$$x_{2i} = -\alpha^{-1} x_i. \tag{9}$$

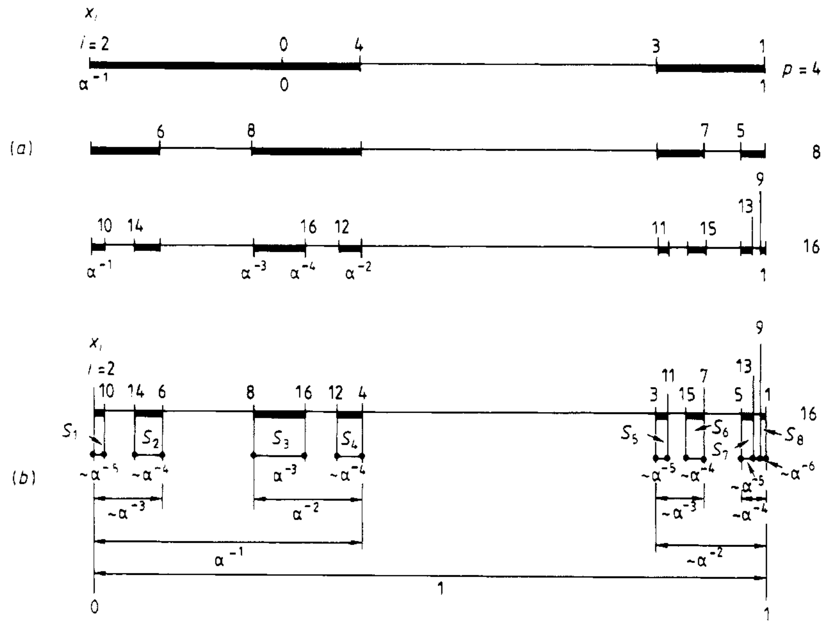

Figure 5. Self-similarity of the Feigenbaum attractor.

For $i=1$, it holds as $x_1=g(0)=1$ by (7), and $x_2=g(1)=-1/\alpha$ by (6). For $i=2$, (9) also holds since $x_4=g(g(x))$ can be rewritten as

$$x_4=-\alpha^{-1}g(-\alpha x_2)=-\alpha^{-1}g(g(0))=-\alpha^{-1}x_2. \tag{10}$$

Now we need to prove that if $x_{2i}=-\alpha^{-1}x_i$, then $x_{2(i+1)}=-\alpha^{-1}x_{i+1}$. In fact,

$$\begin{aligned} x_{2(i+1)} &= g(g(x_{2i}))=-\alpha^{-1}g(-\alpha x_{2i}) \\ &= -\alpha^{-1}g(x_i)=-\alpha^{-1}g(g^{(i)}(0)) \\ &= -\alpha^{-1}x_{i+1}. \end{aligned} \tag{11}$$

Equation (9) is thus proved.

Equation (9) implies that the even subset is just the set $\{x_i\}^{k-1}=\{x_i, i=1,2,\ldots,2^{k-1}\}$ scaled down by a factor $-1/\alpha$, i.e.,

$$\{x_i\}^k_{\text{even}}=\alpha^{-1}\{x_i\}^{k-1}. \tag{12}$$

To see the structure of the odd subset, we notice that the elements of the even subset is generated by applying $g(x)$ once for the corresponding element of the odd subset

$$x_{2i}=g(x_{2i-1}). \tag{13}$$

Since $x_{2i-1}\approx 1$, we can expand $g(x_{2i-1})$ around $x=1$,

$$g(x_{2i-1})\approx g(1)+g'(1)(x_{2i-1}-1). \tag{14}$$

Then (13) becomes

$$x_{2i}-(-1/\alpha)\approx g'(1)(x_{2i-1}-1). \tag{15}$$

That is, the distance between an element x_{2i} of the even (left) subset and the left boundary $x_2=-1/\alpha$ is approximately proportional to the distance between an element x_{2i-1} of the odd (right) subset and the right boundary point $x=1$. In other words, the odd subset is approximately similar to the even subset scaled by a constant $g'(1)$,

$$\{x_i\}^k_{\text{odd}}\approx\frac{1}{g'(1)}\{x_i\}^k_{\text{even}}. \tag{16}$$

Substituting equation (12) into it, we have

$$\{x_i\}^k_{\text{odd}}=(1/\alpha')\{x_i\}^{k-1} \tag{17}$$

where $\alpha'=-\alpha g'(1)$. Together with (10) and (17), we see that the whole set consists of two subsets: the even or left subset which is exactly similar to it, and the odd or right subset which is only approximately so. Therefore,

$$\{x_i\}^k\approx-(1/\alpha)\{x_i\}^{k-1}+(1/\alpha')\{x_i\}^{k-1}. \tag{18}$$

Repeating the same discussion as before, we can divide each subset into two sub-subsets similar to it, and so on. Finally, we conclude that the Feigenbaum attractor can be divided into 2^{m-1} subsets $(m<k)$: $S_1, S_2, \ldots, S_j, \ldots, S_{2^{m-1}}$, each having the same structure but scaled by a different factor $1/s_j$. In this sense, we say the set is self-similar, or Cantor-set-like.

To end the discussion on the even and odd subsets, we give the approximate value for α' below. Since the Feigenbaum invariant function $g(x) = 1 + c_1x^2 + c_2x^4 + \ldots$, and $g(1) = -1/\alpha$, we approximate it up to $O(x^2)$ as

$$g(x) \approx 1 - (1 + 1/\alpha)x^2. \tag{19}$$

Hence, $x_0 = 0$, $x_1 = 1$, $x_2 = -1/\alpha$, $x_3 = g(-1/\alpha) \approx 1 - (1 + 1/\alpha)/\alpha^2$ and $x_4 = 1/\alpha^2$. From (15), we have

$$g'(1) \approx \frac{x_4 - x_2}{x_3 - x_1} \approx -\alpha \tag{20}$$

and

$$\alpha' = -\alpha g'(1) \approx \alpha^2. \tag{21}$$

3.1. *Capacity D*

We first derive the approximate analytical formula for the capacity D. As shown in figure 5(b), the whole set consists of R similar subsets S_j, $j = 1, \ldots, R$. Let s_j denote the scaling factor of the jth subset, then

$$s_j = \frac{1}{x'_{2j} - x'_{2j-1}} \tag{22}$$

where x'_1 is an element of the set $\{x'_i, i = 1, \ldots, R\}$ and is ordered such that

$$0 = x'_1 < x'_2 < x'_3 < \ldots < x'_R = 1. \tag{23}$$

When one uses a bin of size ε to cover the whole attractor, the number of non-empty bins is

$$M(\varepsilon) = \sum_{j=1}^{R} M^{(j)}(\varepsilon) \tag{24}$$

where $M^{(j)}(\varepsilon)$ is the number of bins with length ε needed to cover the jth subset S_j with length $1/s_j$. From the definitions of $M(\varepsilon)$ and D, one has

$$M(\varepsilon) = s_j^{-D} M(\varepsilon/s_j) \tag{25}$$

and, if one considers the jth subset only

$$M^{(j)}(\varepsilon) = s_j^{-D} M^{(j)}(\varepsilon/s_j). \tag{26}$$

Due to self-similarity, one has

$$M^{(j)}(\varepsilon/s_j) = M(\varepsilon). \tag{27}$$

Therefore, using (24)-(27), one obtains

$$M(\varepsilon) = \sum_{j=1}^{R} s_j^{-D} M(\varepsilon). \tag{28}$$

In other words,

$$\sum_{j=1}^{R} s_j^{-D} = 1. \tag{29}$$

This is an implicit equation for D.

3.2. Information dimension σ

Now we turn to the derivation of an approximate formula for the information dimension σ. When one uses a bin of size ε to cover the whole set, one divides the set into $1/\varepsilon$ intervals labelled by k, $k=1, 2, \ldots, 1/\varepsilon$. Let $p_k(\varepsilon)$ be the probability for points to fall into the kth interval, and $p_k^{(j)}(\varepsilon)$ be the probability for points of the jth subset S to fall into the kth interval. Due to self-similarity, one has

$$p_k(\varepsilon)=Rp_k^{(j)}(\varepsilon/s_j). \tag{30}$$

The information entropy is defined as

$$S(\varepsilon)=-\sum_{k=1}^{M(\varepsilon)} p_k(\varepsilon)\ln p_k(\varepsilon). \tag{31}$$

Since the set contains R similar subsets, it can be rewritten as

$$S(\varepsilon)=-\sum_{k=1}^{M(\varepsilon)}\sum_{j=1}^{R} p_k^{(j)}(\varepsilon)\ln(p_k^{(j)}(\varepsilon)). \tag{32}$$

Substituting (30) into it, one has

$$\begin{aligned} S(\varepsilon) &= -\sum_{j=1}^{R}\sum_{k=1}^{M(\varepsilon s_j)}\frac{1}{R}p_k\left(\frac{1}{\varepsilon s_j}\right)\ln\left[\frac{1}{R}p_k\left(\frac{1}{\varepsilon s_j}\right)\right] \\ &= -\sum_{j=1}^{R}\left(\frac{1}{R}\ln\frac{1}{R}\sum_{k=1}^{M(\varepsilon s_j)}p_k(\varepsilon s_j)+\frac{1}{R}\sum_{k=1}^{M(\varepsilon s_j)}p_k(\varepsilon s_j)\ln p_k(\varepsilon s_j)\right) \\ &= -\sum_{j=1}^{R}\frac{1}{R}\ln\frac{1}{R}+\frac{1}{R}\sum_{j=1}^{R}S(\varepsilon s_j). \end{aligned} \tag{33}$$

This is the equation for $S(\varepsilon)$. Its solution is of the form

$$S(\varepsilon)=\sigma\ln(1/\varepsilon). \tag{34}$$

Substituting it into (33), one has

$$\sigma\ln\frac{1}{\varepsilon}=-\sum_{j=1}^{R}\frac{1}{R}\ln\frac{1}{R}+\frac{1}{R}\sum_{j=1}^{R}\sigma\ln\frac{1}{\varepsilon s_j} \tag{35}$$

or

$$\sigma=\frac{R\ln R}{\Sigma_{j=1}^{R}\ln s_j}. \tag{36}$$

This formula gives an analytical approximation for σ.

3.3. Correlation exponent ν

An approximate formula for the correlation exponent ν can be derived in a similar way. The correlation sum $C(\varepsilon)$ of (4) can be approximately expressed as

$$C(\varepsilon)=\sum_{k=1}^{M(\varepsilon)}(p_k(\varepsilon))^2. \tag{37}$$

One then has

$$C(\varepsilon)=\sum_{k=1}^{M(\varepsilon)}\sum_{j=1}^{R}(p_k^{(j)}(\varepsilon))^2$$
$$=\sum_{j=1}^{R}\sum_{k=1}^{M(\varepsilon s_j)}\left(\frac{1}{R}p_k(\varepsilon s_j)\right)^2$$
$$=(1/R)^2\sum_{j=1}^{R}C(\varepsilon s_j). \tag{38}$$

This equation has a solution of the form given by (4). Substituting this form into (38), one has

$$C_0\varepsilon^{\nu}=(1/R)^2\sum_{j=1}^{R}C_0(\varepsilon s_j)^{\nu} \tag{39}$$

or

$$\sum_{j=1}^{R}s_j^{\nu}=R^2. \tag{40}$$

This gives an implicit equation for ν.

The equations (29), (36) and (40) are the approximate analytical formulae for D, σ and ν respectively. To calculate the three dimensions by these equations, we first obtain the strange attractors generated by the Feigenbaum invariant function with different z. For $z=2, 3, 4$ and 5, the results are listed in table 1. Up to sixteen ($R=16$) subsets of the attractor have been used. One can see that the analytical approximations agree quite well with the numerical results calculated directly from the definitions.

4. Concluding remarks

It is clear from figure 4 that, although the rigorous inequality $D\geqslant\sigma\geqslant\nu$ is always obeyed, the variation of D, σ and ν with z is very different. D seems to increase with z; σ seems to increase to a more or less constant value very quickly; and ν seems to increase to a certain value and then decrease. Therefore it is important to emphasise that, although the capacity, the information dimension and the correlation exponent are sometimes all branded as 'fractal dimension', they are actually different concepts and should not be used interchangeably for the sake of expedience. Their behaviour can be very different as the control parameter is changed.

It is nowadays fashionable to invent, as a matter of expediency, new definitions of dimension. Useful as these definitions may be, they might not have anything to do with the original concept of dimension. In this sense the word 'dimension' may have been abused. It may be instructive to learn from a closely related field: critical phenomena. In critical phenomena there are nine critical exponents. They are related, and they satisfy various inequalities; however, they denote different physical quantities. One exponent may be more difficult to calculate or measure than the other; however, it is not proper to use one as a substitute for the other.

Finally, a word as to the physical relevance of the degeneracies of the critical point for the cases other than $z=2$. Indeed, present experiments are largely consistent with the quadratic case. However, as pointed out by Kuramoto and Koga (1982), other

z values might arise in chemical turbulence. It is also entirely conceivable, for symmetry or other reasons, for the quadratic term to vanish and higher z to come into play. It is therefore premature to preclude the physical relevance of the other degeneracies of the critical point.

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